

Real-Time Computability of Real Numbers by Chemical Reaction Networks ^{*}

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Abstract. We explore the class of real numbers that are computed in real time by deterministic chemical reaction networks that are *integral* in the sense that all their reaction rate constants are positive integers. We say that such a reaction network *computes* a real number α in *real time* if it has a designated species X such that, when all species concentrations are set to zero at time $t = 0$, the concentration $x(t)$ of X is within 2^{-t} of the fractional part of α at all times $t \geq 1$, and the concentrations of all other species are bounded. We show that every algebraic number is real time computable by chemical reaction networks in this sense. We discuss possible implications of this for the 1965 Hartmanis-Stearns conjecture, which says that no irrational algebraic number is real time computable by a Turing machine.

Keywords: analog computation, chemical reaction networks, Hartmanis-Stearns conjecture, real-time computability

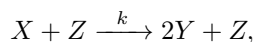
1 Introduction

Chemical reaction networks, originally conceived as descriptive mathematical models of molecular interactions in well-mixed solutions, are also widely used as prescriptive mathematical models for engineering molecular processes. In the present century this prescriptive use of chemical reaction networks has been automated by software compilers that translate chemical reaction networks into complete specifications of DNA strand displacement systems that simulate them [21, 4]. Chemical reaction networks have thus become the programming language of choice for many molecular programming applications.

There are several alternative semantics (operational meanings, also called kinetics) for chemical reaction networks. The two oldest and most widely used of these are *deterministic mass-action semantics* and *stochastic mass-action semantics*. This paper concerns the former of these, so for the rest of this paper, a

^{*} This research was supported in part by National Science Foundation Grants 1247051 and 1545028.

chemical reaction network (briefly, a *CRN* or a *deterministic CRN*) is a chemical reaction network with deterministic mass-action semantics. This model is precisely specified in section 2 below. For this introduction, it suffices to say that such a CRN is an ordered pair $N = (S, R)$, where S is a finite set of *species* (abstract molecule types), and R is a finite set of *reactions*, each of which has some form like:



where $X, Y, Z \in S$ are species and $k \in [0, \infty)$ is a *rate constant*. A *state* \mathbf{x} of N specifies the real-valued *concentration* $\mathbf{x}(Y) \in [0, \infty)$ of each species Y . Given an initial state $\mathbf{x}(0)$ at time $t = 0$, deterministic mass action semantics specify the (continuous) evolution of the state $\mathbf{x}(t)$ over time.

Even prior to the implementation of chemical reaction networks as a programming language it was clear that they constitute a model of computation. In the case of deterministic CRNs, Stansifer has reportedly proven [5] that this model is Turing universal, i.e., that every algorithm can be simulated by a deterministic CRN. (Note: The title of [17] seems to make this assertion, but the paper only exhibits a way to use deterministic CRNs to simulate finite Boolean circuits.)

Deterministic chemical reaction networks are an analog model of computation, both in the intuitive sense that their states are vectors of real-valued concentrations that vary continuously over real-valued times and in the technical sense that they are a special case of Shannon's *general purpose analog computer (GPAC)* [20], as explained in section 5 below.

This paper studies the ability of deterministic CRNs to rapidly compute real numbers in the following analog sense. We say that a deterministic CRN *computes* a real number α *in real time* if it has a designated species X such that the following three things hold. (See section 3 for more details.) First, the CRN's reaction rate constants are positive integers, and it is initialized with all concentrations set to zero at time $t = 0$. This implies that the CRN is, like any reasonable model of computation, finitely specifiable. It also implies that only countably many real numbers are real time CRN-computable. Second, there is some fixed bound on all the CRN's concentrations. Under deterministic mass-action semantics, this implies that all the reaction rates of the CRN are bounded, whence time is a meaningful resource. Third, the concentration $x(t)$ of the designated species $X(t)$ is within 2^{-t} of the fractional part $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ of α – i.e., within t bits of accuracy of $\{\alpha\}$ – at all times $t \geq 1$. We say that the real number α is *real time computable by chemical reaction networks* (briefly, *real time CRN-computable*) if there is a CRN that computes α in this sense. Elementary properties of real-time CRN computability are developed in section 3.

Our main theorem says that every algebraic number (i.e., every real solution of a polynomial with integer coefficients) is real time CRN-computable. This result is proven in section 4. We conjecture, but have not proven at the time of this writing, that some transcendental (i.e., non-algebraic) real numbers are also real time CRN-computable.

Our main theorem is a counterpoint – but not a disproof – of the 57-year-old, open Hartmanis-Stearns conjecture that no algebraic irrational is real time

computable by a Turing machine [12]. Section 5 discusses this contrast in some detail and poses two questions whose answers would shed further light on the computational complexities of algebraic irrationals.

2 Chemical Reaction Networks

A *species* is an abstract type of molecule. Capital Roman characters such as X , Y , and Z are commonly used to distinguish different species, but we also use decorations such as X_0 , \hat{Y} , and \bar{Z} to distinguish them.

A *reaction* over a finite set S of species is a tuple $\rho = (\mathbf{r}, \mathbf{p}, k) \in \mathbb{N}^S \times \mathbb{N}^S \times (0, \infty)$ and its components are called the *reactant vector*, the *product vector*, and the *rate constant*, respectively. (Here \mathbb{N}^S denotes the set of all functions mapping S into \mathbb{N} .) To avoid excessive use of subscripts, for a reaction ρ we use $\mathbf{r}(\rho)$, $\mathbf{p}(\rho)$, and $k(\rho)$ to access the individual components of ρ . A species $Y \in S$ is called a *reactant* if $\mathbf{r}(Y) > 0$, called a *product* if $\mathbf{p}(Y) > 0$, and called a *catalyst* if $\mathbf{r}(Y) = \mathbf{p}(Y) > 0$. The *net effect* of reaction $\rho = (\mathbf{r}, \mathbf{p}, k)$ is the vector $\Delta\rho \in \mathbb{N}^S$ defined by

$$\Delta\rho(Y) = \mathbf{p}(Y) - \mathbf{r}(Y)$$

for each $Y \in S$.

A *chemical reaction network (CRN)* is an ordered pair $N = (S, R)$ where S is a finite set of species and R is a finite set of reactions over S . Although this completes the definition of the *syntax* of a CRN, we have yet to define the *semantics* used in this paper.

Under *deterministic mass action semantics*, the *state* of a CRN $N = (S, R)$ at time t is a real-valued vector $\mathbf{x}(t) \in [0, \infty)^S$, and for $Y \in S$, we call $\mathbf{x}(t)(Y)$ the *concentration* of Y in $\mathbf{x}(t)$. We also write $y(t) = \mathbf{x}(t)(Y)$ to denote the concentration of species Y at time t .

The *rate* of a reaction ρ at time t is defined as

$$\text{rate}_\rho(t) = k(\rho) \cdot \prod_{Y \in S} y(t)^{\mathbf{r}(\rho)(Y)}. \quad (2.1)$$

This conforms to the so-called law of mass action which states that the rate of a reaction is proportional to the concentration of its reactants.

The *total rate of change* of a species $Y \in S$ depends on the rates of all reactions in the CRN and the magnitude of their net effect on Y . Therefore the concentration $y(t)$ conforms to the ordinary differential equation (ODE)

$$\frac{dy}{dt} = \sum_{\rho \in R} \Delta\rho(Y) \cdot \text{rate}_\rho(t) \quad (2.2)$$

If we let \mathcal{E}_Y be the ODE above for each $Y \in S$, then the *mass action system* of the CRN is the coupled system

$$(\mathcal{E}_Y \mid Y \in S). \quad (2.3)$$

Given an initial state $\mathbf{x}_0 \in [0, \infty)^S$, the behavior of the CRN is defined as the solution to the initial value problem (IVP) of the mass action system (2.3) along with the initial condition

$$y(0) = \mathbf{x}_0(Y)$$

for each $Y \in S$.

3 Real-Time CRN Computability

We say that a real number α is *real time computable by chemical reaction networks* (briefly, *real time CRN-computable*), and we write $\alpha \in \mathbb{R}_{RTCRN}$, if there exist a chemical reaction network $N = (S, R)$ and a species $X \in S$ with the following three properties:

1. (integrality). The CRN N is *integral* in the sense that:

$$k(\rho) \in \mathbb{Z}^+ \tag{3.1}$$

for all $\rho \in R$.

2. (boundedness). There is a constant $\beta > 0$ such that, if N is initialized with $y(0) = 0$ for all $Y \in S$, then, for all $Y \in S$ and $t \in [0, \infty)$,

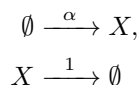
$$y(t) \leq \beta. \tag{3.2}$$

3. (real-time convergence). If N is initialized with $y(0) = 0$ for all $Y \in S$, then for all $t \in [1, \infty)$,

$$|x(t) - \{\alpha\}| \leq 2^{-t} \tag{3.3}$$

where $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ is the *fractional part* of α .

The integrality condition (3.1) prevents the CRN N from “cheating” by having information about α explicitly encoded into its rate constants. To see that this is necessary to avoid nontriviality, note that, for any $\alpha \in (0, 1)$, if the simple CRN:



is initialized with $x(0) = 0$, then

$$x(t) = \alpha(1 - e^{-t})$$

for all $t \in [0, \infty)$.

The boundedness condition (3.2) imposes a “speed limit” on the CRN N . This prevents N from acting as a “Zeno machine” (machine that does infinite work in finite time) in the sense of Weyl [26]. More precisely, condition (3.2) ensures that the reaction rates (2.1) of N are all bounded. This implies that the arc length of the curve traced by the state $\mathbf{x}(s)$ of N for $0 \leq s \leq t$ is $\theta(t)$, i.e., bounded above and below by positive constant multiples of t . Pouly [19, 1] has convincingly

argued (in a more general setting) that this arc length, which we call the *reaction clock time*, is the correct measure of the time that a CRN spends computing during the interval $[0, t]$. Viewed in this light, condition (3.2) ensures that t is, up to constant multiples, an accurate measure of the reaction clock time of N during the interval $[0, t]$.

The real-time convergence condition (3.3) requires the CRN N to compute $\{\alpha\}$ to within t bits of accuracy by each time $t \geq 1$. Note that this is an *analog* approximation of $\{\alpha\}$. The CRN N is not required to explicitly produce symbols in any sort of digital representation of $\{\alpha\}$.

For the rest of this paper, unless otherwise noted, all CRNs $N = (S, R)$ are assumed to be initialized with $y(0) = 0$ for all $Y \in S$.

To save space in our first lemma, we define the predicate

$$\begin{aligned} \Phi_{\tau, \gamma}(\alpha) \equiv & \text{there exist a CRN } N = (S, R) \text{ and a species } X \in S \\ & \text{satisfying (3.1) and (3.2) such that, for all } t \in [\tau, \infty), \\ & |x(t) - \{\alpha\}| \leq e^{-\gamma t} \end{aligned}$$

for each $\tau, \gamma \in (0, \infty)$ and $\alpha \in \mathbb{R}$. Note that $\Phi_{1, \ln 2}(\alpha)$ is the assertion that $\alpha \in \mathbb{R}_{RTCRN}$. The following convenient lemma says that the definition of \mathbb{R}_{RTCRN} is robust with respect to linear changes in condition (3.2).

Lemma 3.1. *For each $\alpha \in \mathbb{R}$ the following conditions are equivalent.*

1. $\alpha \in \mathbb{R}_{RTCRN}$.
2. There exists $\tau, \gamma \in (0, \infty)$ such that $\Phi_{\tau, \gamma}(\alpha)$ holds.
3. For every $\tau, \gamma \in (0, \infty)$, $\Phi_{\tau, \gamma}(\alpha)$ holds.

Proof. Let $\alpha \in \mathbb{R}$. It is clear that (3) \Rightarrow (1) \Rightarrow (2), so it suffices to prove that (2) \Rightarrow (3). For this, let N, X, τ , and γ testify that (2) holds, i.e., let N and X testify that $\Phi_{\tau, \gamma}(\alpha)$ holds. To prove (3), let $\hat{\tau}, \hat{\gamma} \in (0, \infty)$. It suffices to show that $\Phi_{\hat{\tau}, \hat{\gamma}}(\alpha)$ holds. Let

$$a = \max \left\{ \left\lceil \frac{\tau}{\hat{\tau}} \right\rceil, \left\lceil \frac{\hat{\gamma}}{\gamma} \right\rceil \right\},$$

and let $\hat{N} = (S, \hat{R})$, where

$$\hat{R} = \{(\mathbf{r}, \mathbf{p}, ak) \mid (\mathbf{r}, \mathbf{p}, k) \in R\}.$$

That is, \hat{N} is exactly like N , except that each rate constant of N has been multiplied by the positive integer a . Then \hat{N} is an integral CRN that is a “sped up version” of N in the sense that, for all $y \in S$ and $t \in [0, \infty)$,

$$y_{\hat{N}}(t) = y_N(at), \tag{3.4}$$

where y_N and $y_{\hat{N}}$ are the values of y in N and \hat{N} , respectively. This immediately implies that \hat{N} satisfies (3.2). Now let $t \in [\hat{\tau}, \infty)$. Then $at \in [\tau, \infty)$, so our

assumption $\Phi_{\tau,\gamma}(\alpha)$ tells us that

$$\begin{aligned} |x_{\widehat{N}}(t) - \{\alpha\}| &= |x_N(at) - \{\alpha\}| \\ &\leq e^{-\gamma at} \\ &\leq e^{-\widehat{\gamma}t}, \end{aligned}$$

affirming $\Phi_{\widehat{\tau},\widehat{\gamma}}(\alpha)$. □

The following lemma is a warm-up for our examination of \mathbb{R}_{RTCRN}

Lemma 3.2. $\mathbb{Q} \subsetneq \mathbb{R}_{RTCRN}$

Proof. If $\alpha \in \mathbb{Z}$, then the CRN

$$X \xrightarrow{1} \emptyset$$

satisfies

$$|x(t) - \{\alpha\}| = x(t) = e^{-t} \leq 2^{-t},$$

so $\alpha \in \mathbb{R}_{RTCRN}$. If $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, then we can write $\{\alpha\} = \frac{a}{b}$, where $a, b \in \mathbb{Z}^+$. Then the integral CRN

$$\begin{aligned} \emptyset &\xrightarrow{a} X \\ X &\xrightarrow{b} \emptyset \end{aligned}$$

satisfies

$$x(t) = \frac{a}{b}(1 - e^{-bt}),$$

so $\alpha \in \mathbb{R}_{RTCRN}$ by Lemma 3.1. This shows that $\mathbb{Q} \subseteq \mathbb{R}_{RTCRN}$.

To see that $\mathbb{Q} \neq \mathbb{R}_{RTCRN}$, it suffices to show that $\frac{1}{\sqrt{2}} \in \mathbb{R}_{RTCRN}$. Since the integral CRN

$$\begin{aligned} \emptyset &\xrightarrow{1} X \\ 2X &\xrightarrow{2} X \end{aligned}$$

satisfies

$$x(t) = \frac{1}{\sqrt{2}} \left(\frac{1 - e^{-2\sqrt{2}t}}{1 + e^{-2\sqrt{2}t}} \right),$$

we have that

$$\begin{aligned} \left| x(t) - \frac{1}{\sqrt{2}} \right| &= \frac{1}{\sqrt{2}} \left(\frac{e^{-2\sqrt{2}t}}{1 + e^{-2\sqrt{2}t}} \right) \\ &\leq \frac{1}{\sqrt{2}} e^{-2\sqrt{2}t} < e^{-2\sqrt{2}t}, \end{aligned}$$

so $\frac{1}{\sqrt{2}} \in \mathbb{R}_{RTCRN}$ by Lemma 3.1. □

Computable real numbers were introduced by Turing [23, 24] and have been extensively investigated [14, 25].

A real number α is *computable*, and we write $\alpha \in \mathbb{R}_{comp}$, if there is a computable function $\hat{\alpha} : \mathbb{N} \rightarrow \mathbb{Q}$ such that, for all $r \in \mathbb{N}$

$$|\hat{\alpha}(r) - \alpha| \leq 2^{-r}.$$

Lemma 3.3. $\mathbb{R}_{RTCRN} \subsetneq \mathbb{R}_{comp}$

Proof. Let $\alpha \in \mathbb{R}_{RTCRN}$, and let $N = (S, R)$ and $X \in S$ testify to this fact. Let Y_1, \dots, Y_n be the distinct species in S . Then the ODEs (2.2) can be written in the form

$$\begin{aligned} y_1' &= f_1(y_1, \dots, y_n), \\ &\vdots \\ y_n' &= f_n(y_1, \dots, y_n), \end{aligned} \tag{3.5}$$

where f_1, \dots, f_n are polynomials with integer coefficients. By the boundedness condition (3.2) and Theorem 16 of [8], the solution $\mathbf{y} : [0, \infty) \rightarrow [0, \infty)^n$ of (3.5) is polynomial time computable. It follows by the real-time convergence condition (3.3) that α is computable in polynomial time in the sense of Ko [14]. Hence, $\alpha \in \mathbb{R}_{comp}$.

It is well known [14] that not every computable real is computable in polynomial time, so the preceding paragraph proves the lemma. \square

4 Algebraic Numbers Are Real Time CRN Computable

This section is devoted to proving the following result, which is our main theorem.

Theorem 4.1. *Every algebraic number is an element of \mathbb{R}_{RTCRN} .*

Our proof of Theorem 4.1 uses the stability theory of ordinary differential equations. We review the elements of this theory that we need here, referring the reader to standard textbooks (e.g., [13, 22]) for more thorough treatments.

We first note that the ordinary differential equations (2.2) of a CRN $N = (S, R)$ are *autonomous*, meaning that they only depend on the time t via the species concentrations $y(t)$. Hence, if we let Y_1, \dots, Y_n be the distinct species in S , then the ODEs (2.2) can be written as

$$\begin{aligned} y_1' &= f_1(y_1, \dots, y_n), \\ &\vdots \\ y_n' &= f_n(y_1, \dots, y_n), \end{aligned} \tag{4.1}$$

where $f_1, \dots, f_n : \mathbb{R}^n \rightarrow \mathbb{R}$ are polynomials. If we let $\mathbf{f}_N : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the function whose components are f_1, \dots, f_n , then (4.1) can be written in the vector form

$$\mathbf{x}' = \mathbf{f}_N(\mathbf{x}). \tag{4.2}$$

The *Jacobian matrix* of the CRN N is the Jacobian matrix of \mathbf{f}_N , i.e., the $n \times n$ matrix

$$J_N = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \dots & \frac{\partial f_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \dots & \frac{\partial f_n}{\partial y_n} \end{pmatrix}.$$

More precisely, the *Jacobian matrix* of N in a state $\mathbf{x} \in [0, \infty)^S$ is the matrix $J_N(\mathbf{x})$ in which each of the partial derivatives in J_N is evaluated at the point \mathbf{x} . The *eigenvalues* of the CRN N in a state $\mathbf{x} \in [0, \infty)^S$ are the eigenvalues of the matrix $J_N(\mathbf{x})$, i.e., the numbers $\lambda \in \mathbb{C}$ for which there exists $\mathbf{y} \in \mathbb{R}^n$ such that $J_N(\mathbf{x})(\mathbf{y}) = \lambda\mathbf{y}$.

A *fixed point* of the CRN N is a state $\mathbf{z} \in [0, \infty)^S$ such that $\mathbf{f}_N(\mathbf{z}) = \mathbf{0}$. A fixed point \mathbf{z} of N is *exponentially stable* if there exist $\alpha, \delta, C \in (0, \infty)$ such that, for all $\mathbf{x}_0 \in [0, \infty)^S$ with $|\mathbf{x}_0 - \mathbf{z}| \leq \delta$, if N is initialized with $\mathbf{x}(0) = \mathbf{x}_0$, then, for all $t \in [0, \infty)$,

$$|\mathbf{x}(t) - \mathbf{z}| \leq Ce^{-\alpha t} |\mathbf{x}(0) - \mathbf{z}|. \quad (4.3)$$

The well known *exponential stability theorem*, specialized to CRNs, says that a fixed point \mathbf{z} of N is exponentially stable if all its eigenvalues have negative real parts [13, 22].

In this paper we define a real number $\alpha \in \mathbb{R}$ to be *negative eigenvalue computable by chemical reaction networks* (briefly, *negative eigenvalue CRN-computable*), and we write $\alpha \in \mathbb{R}_{NECRN}$, if there exist a CRN $N = (S, R)$, a species $X \in S$, and a state $\mathbf{z} \in [0, \infty)^S$ with $\mathbf{z}(X) = \alpha$ such that the following conditions hold.

1. (integrality). The CRN N is integral as in (3.1).
2. (boundedness). Concentrations are bounded as in (3.2).
3. (fixed point). \mathbf{z} is a fixed point of N .
4. (negative eigenvalues). All the eigenvalues of N in the state \mathbf{z} have negative real parts.
5. (basin of attraction). If α, δ , and C are the constants testifying that \mathbf{z} is exponentially stable, then the zero-vector $\mathbf{0} \in [0, \infty)^S$ defined by $\mathbf{0}(Y) = 0$ for all $Y \in S$ satisfies $|\mathbf{0} - \mathbf{z}| \leq \delta$.

Our interest in the class \mathbb{R}_{NECRN} is that the following three lemmas suffice to prove Theorem 4.1.

Lemma 4.2. \mathbb{R}_{NECRN} is a countable subfield of \mathbb{R} .

Lemma 4.3. $\mathbb{R}_{NECRN} \subseteq \mathbb{R}_{RTCRN}$.

Proof. Let $\alpha \in \mathbb{R}_{NECRN}$. We show in the full version of this paper that $\alpha - \lfloor \alpha \rfloor$ is also in \mathbb{R}_{NECRN} . Without loss of generality, we assume that $\alpha \in (0, 1)$. Hence we have $\alpha = \{\alpha\}$ in the following proof.

By the definition of \mathbb{R}_{NECRN} , there is a CRN $N = (S, R)$, a species $X \in S$, and a state $\mathbf{z} \in [0, \infty)^S$ with $\mathbf{z}(X) = \alpha$ such that $\mathbf{0}$ falls in the basin of attraction of \mathbf{z} . Therefore $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{z}$.

Since $J_N(\mathbf{z})$ has eigenvalues with negative real parts, then \mathbf{z} is exponentially stable, i.e. there exist $\alpha, \delta, C \in (0, \infty)$ such that for all $\mathbf{x}_0 \in [0, \infty)^S$ with $|\mathbf{x}_0 - \mathbf{z}| \leq \delta$, if N is initialized with $\mathbf{x}(0) = \mathbf{x}_0$, then for all $t \in [0, \infty)$, $|\mathbf{x}(t) - \mathbf{z}| \leq Ce^{-\alpha t} |\mathbf{x}_0 - \mathbf{z}|$.

Consider the CRN N initialized so that $\mathbf{x}(0) = \mathbf{0}$. Since $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{z}$, we let τ_0 be the point such that $|\mathbf{x}(\tau_0) - \mathbf{z}| \leq \delta$, then by exponential stability of \mathbf{z} , we have $|\mathbf{x}(t) - \mathbf{z}| \leq Ce^{-\alpha(t-\tau_0)} |\mathbf{x}_0 - \mathbf{z}|$ for all $t \geq \tau_0$.

Pick a number γ such that,

$$Ce^{\alpha\tau_0} |\mathbf{x}_0 - \mathbf{z}| \leq e^{\gamma\tau_0}$$

and let $a = \left\lceil \frac{2\gamma}{\alpha} \right\rceil$, construct a ‘‘sped up version’’ of N , \hat{N} , as in Lemma 3.1, by multiplying each rate constant of N by the positive integer a . Now let $\tau = \frac{\tau_0}{a}$. Then for all $t \geq \tau$, i.e., $at \geq \tau_0$, we have

$$\begin{aligned} |\mathbf{x}_{\hat{N}}(X)(t) - \alpha| &\leq |\mathbf{x}_{\hat{N}}(t) - \mathbf{z}| \\ &= |\mathbf{x}(at) - \mathbf{z}| \\ &\leq Ce^{-\alpha(at-\tau_0)} |\mathbf{x}_0 - \mathbf{z}|, \text{ since } at \geq \tau_0 \\ &\leq e^{\gamma\tau_0} e^{-a\alpha t} \\ &\leq e^{\gamma\tau_0} e^{-2\gamma t} \\ &\leq e^{-\gamma t} \end{aligned}$$

Hence $\Phi_{\tau, \gamma}(\alpha)$ holds, and by Lemma 3.1, $\alpha \in \mathbb{R}_{RTCRN}$. □

Lemma 4.4. *Every algebraic number is an element of \mathbb{R}_{NECRN} .*

5 Discussion

We have shown that every algebraic number is real time computable by deterministic chemical reaction networks. What does this say about the complexity of algebraic irrationals on other models of computation?

The first thing to understand here is that deterministic chemical reaction networks are, in a very precise sense, a model of analog computation. In 1941, Shannon [Shan41] introduced the *general-purpose analog computer (GPAC)*. A GPAC is a mathematical abstraction of the *differential analyzer*, an early analog computer that Bush [3] had constructed at MIT, and which Shannon had operated as a graduate research assistant. The GPAC model has been corrected and otherwise modified a number of times over the years [18, 15, 7, 9]. Its present form can be characterized in terms of circuits, but it is more simply characterized as a system

$$\mathbf{y}'(t) = \mathbf{p}(t, \mathbf{y}), \tag{5.1}$$

of ordinary differential equations, where \mathbf{p} is a vector of polynomials. A deterministic CRN is thus a special type of GPAC of the form

$$\mathbf{y}'(t) = \mathbf{p}(\mathbf{y}), \tag{5.2}$$

where each component p_i of \mathbf{p} has the “kinetic” form $p_i(\mathbf{y}) = q_i(\mathbf{y}) - y_i r_i(\mathbf{y})$, with q_i and r_i having nonnegative coefficients [11]. Our CRNs in this paper have the added constraints that all the coefficients in these polynomials are integers, and all concentrations are initialized to zero. Our main theorem thus implies that all algebraic numbers are real time computable by GPACs that have only finite information coded into their parameters and initializations.

We now turn from analog computation to discrete computation. A famous conjecture of Hartmanis and Stearns [12] says that no irrational algebraic number is real time computable by a Turing machine. This conjecture has been open for over 50 years. Fischer, Meyer, and Rosenberg [6] proved that real-time computability on a Turing machine is equivalent to linear-time computability on a Turing machine. Hence the Hartmanis-Stearns conjecture is equivalent to the statement that no irrational algebraic number is linear-time computable by a Turing machine. As observed by Gurevich and Shelah [10], linear time is a very model-dependent notion. Hence, as stated, the Hartmanis-Stearns conjecture is a very specific conjecture about linear-time computation on Turing machines.

Our main theorem does not disprove the Hartmanis-Stearns conjecture (nor was it intended to), but conceptually locating the gap between our main theorem and a disproof of the Hartmanis-Stearns conjecture would shed light on the computational complexities of algebraic irrationals. This raises the following questions.

Question 1. Can CRNs in our model (or GPACs with only finite information encoded into their parameters and initializations) produce in linear time the individual digits of each real number that is real time CRN-computable? If so, our main theorem implies that the Hartmanis-Stearns conjecture fails for analog computation. If not, the Hartmanis-Stearns conjecture holds for analog computation and is essentially about producing the individual digits as opposed to the analog convergence that we have used here.

Question 2. Is there a reasonable discrete model of computation on which some algebraic irrational can be computed in linear time? If so, then the Hartmanis-Stearns conjecture is either false or model-dependent. If not, then the Hartmanis-Stearns conjecture is true in a strong, model-independent way, at least for discrete computation. (Note that “reasonable” here excludes models that perform numerical operations faster than we know how to do them, because Brent [2] has shown how to compute $\sqrt{2}$ in linear time if integer multiplication can be done in linear time. See also [16].)

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