Problem 0. Document how much time you spend on each of the following problems and cite any resources you received help from.

Problem 1. Recall that a relation $\sim\colon \mathbb{R}\times\mathbb{R}\to\{\text{true},\text{false}\}$ is an *equivalence relation* if it is reflexive, symmetric, and transitive. Recently, I encountered a convincing argument that the reflexive requirement is unnecessary and can be derived from symmetry and transitivity alone. Here is the argument:

Proof. Let $\sim\colon \mathbb{R}\times\mathbb{R}\to\{\text{true},\text{false}\}$ be a relation that is symmetric and transitive. By symmetry, we know that $x\sim y$ if and only if $y\sim x$. By transitivity, we know that if $x\sim y$ and $y\sim z$, then $x\sim z$.

We can obtain reflexivity by using symmetry and transitivity. For any x, by symmetry, we have $x\sim y$ if and only if $y\sim x$. Now we can deploy transitivity and say that since $x\sim y$ and $y\sim x$ that $x\sim x$. Thus, $\sim$ is also reflexive. □

What is the error in the above proof?

Problem 2. Recall that the *power set* of a set A is the set of all *subsets* of A and is denoted $\mathcal{P}(A)$. For example, if $A=\{1,2,3\}$, then

$$\mathcal{P}(A)=\big\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\big\}.$$

Prove or disprove the following statement:

If A is a finite set with cardinality $|A|=n\geq 0$, then the power set of A has cardinality $|\mathcal{P}(A)|=2^n$.

Problem 3. Let $M=(Q,\Sigma,\delta,q_0,F)$ be a DFA where

$$Q=\{q_0,q_1,q_2\},\qquad \Sigma=\{\mathtt{x},\mathtt{y}\},\qquad F=\{q_0\},$$

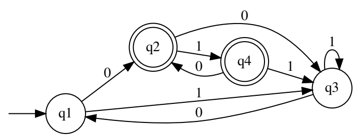
and where δ is defined by the transition table

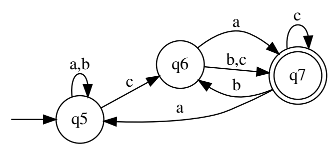
δ	x	y
q_0	q_0	q_1
q_1	q_0	q_2
q_2	q_0	q_0

(a) Draw the visual representation of M.

(b) Describe in your own words what language M recognizes.

Problem 4. Give the formal descriptions of each of the following DFAs (i.e. set of states, transition table, etc.).





Bonus Problem (Extra Credit)**.** Dr. Adam Case challenges you and your friend Jasmine to a friendly three-player game. Before starting the game, Dr. Case first explains all of the rules and gives you both enough time to develop and agree on a strategy.

At the beginning of the game, you and Jasmine will be separated and cannot communicate directly for the entirety of the game. Once you are separated, the game will proceed in three steps:

1. Dr. Case will pick an integer $n \in \{0, 1, 2, \dots, 7\}$ and pick an 8-bit string w (e.g. 10001010). He then shows you both n and w.
2. At this point, you *must* select exactly one bit of w and flip it (i.e. a 0 becomes 1 and vice versa). Let $\widehat{w}$ be this modified string.
3. Dr. Case then shows $\widehat{w}$ to your friend Jasmine, and Jasmine will try to figure out what n is using only $\widehat{w}$. (She never sees the original w.)

If Jasmine can correctly derive the number n from $\widehat{w}$, then you and Jasmine win. However, if Jasmine picks incorrectly, then Dr. Case wins.

(a) Prove that you and Jasmine have a winning strategy. In other words, there exists an agreed-upon strategy that leads to *certain victory* for all choices of n and w that Dr. Case picks.

(b) If the game were generalized so that $n \in \{0, 1, 2, 3, 4, \dots, k-1\}$ and w is k bits long, prove that Dr. Case has a winning strategy for at least one $k > 0$. In other words, for any strategy that you and Jasmine choose, there exists an n and w that Dr. Case can pick that will make you lose.

(c) Prove that you and Jasmine have a winning strategy if and only if k is a power of 2.