Recall that we say a set $S$ is *closed* under an operation if applying that operation to an element of $S$ produces another element of of $S$. For example, the set $\mathbb{N}$ of natural numbers is closed under addition since for any two numbers $n_1, n_2 \in \mathbb{N}$, the number $n_1 + n_2$ is still in $\mathbb{N}$. However, $\mathbb{N}$ is *not* closed under subtraction since $3 - 5 = -2$ is not in $\mathbb{N}$.

**Claim 1.** The class of regular languages is closed under *concatenation* and the *Kleene star*.

*Proof.* Assume that $A_1$ and $A_2$ are regular languages. Therefore there exist regular expressions $R_1$ and $R_2$ that recognize them, respectively. It now suffices to show that $A_1 \circ A_2$ and $A_1^*$ are regular, and we will show this by giving regular expressions that recognize them. Since regular expressions natively support concatenation and Kleene star operations, we know that $(R_1 \circ R_2)$ and $(R_1)^*$ are also proper regular expressions. By the definition of $\circ$ and $*$ in regular expressions, we know that these recognize $A_1 \circ A_2$ and $A_1^*$, respectively, and so they are regular. □

**Claim 2.** The class of regular languages is closed under set complement.

*Proof.* Let $A \subseteq \Sigma^*$ be a regular language over alphabet $\Sigma$, and let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA that testifies to this. It now suffices to show that $\overline{A} = \Sigma^* \setminus A$ is also regular, and we will show this by building a DFA to recognize $\overline{A}$.

Notice that by the definition of set complement, we know that for any string $w \in \Sigma^*$ the following must hold:

$$w \in A \iff w \notin \overline{A}.$$

Therefore it suffices to build a machine that *rejects* all the strings that $M$ accepts and *accepts* all of the strings that $M$ rejects. Therefore let $\overline{M} = (Q, \Sigma, \delta, q_0, \overline{F})$ be a DFA that is equivalent to $M$ other than the set of accepting states is now $\overline{F} = Q \setminus F$. This means that all the reject states of $M$ are accepts states in $\overline{M}$ and all the accepts states of $M$ are now reject states in $\overline{M}$. This means that a string $w \in \Sigma^*$ is accepted by $\overline{M}$ if and only if it is rejected by $M$. Thus $\overline{M}$ recognizes $\overline{A}$, and so it is regular. □

**Claim 3.** All languages with finitely many strings are regular.

*Proof.* Let $A = \{w_1, w_2, \ldots, w_n\}$ be a finite language with $n$ elements. It now suffices to show that $A$ is regular, and we will do this by building a regular expression for the language $A$. For each $w_i \in A$, we know that $w_i$ is itself a valid regular expression. For example, the regular expression 0110 recognizes the language $\{0110\}$. We also know that given two strings $w_i$ and $w_j$, $w_i \cup w_j$ is a valid regular expression that recognizes $\{w_i, w_j\}$. Therefore the following regular expression will recognize the language $A$:

$$R = (w_1 \cup w_2 \cup w_3 \cdots \cup w_n).$$

We know that $R$ is a valid regular expression because it only consists of symbols, concatenation, and union operations. It is also clear the $R$ will only recognize the strings in $A$ that are "hard coded" into the expression, so $A$ must be regular. □

**Claim 4.** The set of *non-regular* languages is closed under set complement.

*Proof.* The statement is equivalent to the implication:

$$A \text{ is not regular} \implies \overline{A} \text{ is not regular}.$$

For convenience, we can change this implication into its equivalent contrapositive form:

$$\overline{A} \text{ is regular} \implies A \text{ is regular}.$$

This is equivalent to the claim that "the set of regular languages is closed under set complement" which we have already proven previously, therefore we know that the set of non-regular languages is also closed under set complement. □