Claim 1. Let $\sim: \mathbb{R} \times \mathbb{R} \to \{\text{true}, \text{false}\}$ be the binary relation defined by

$$a \sim b \iff a - b \in \mathbb{Z}.$$

Prove or disprove: $\sim$ is an equivalence relation.

Proof. We prove the statement is *true* by showing that $\sim$ is reflexive, symmetric, and transitive.

To see that $\sim$ is reflexive, we need to show that for each $x \in \mathbb{R}$ that $x \sim x$ holds. To show this, we let $x \in \mathbb{R}$ be an arbitrary element. Since $x - x = 0 \in \mathbb{Z}$, we know that $x \sim x$, and therefore $\sim$ is reflexive.

To see that $\sim$ is symmetric, we need to show that if $x \sim y$, then $y \sim x$. To show this, let $x, y \in \mathbb{R}$ be arbitrary elements such that $x \sim y$. By the definition of $\sim$, this means that $(x - y) = n \in \mathbb{Z}$. Since n is an integer, we know that $-n = (y - x) \in \mathbb{Z}$ is also an integer. Therefore we know that $y \sim x$, and therefore $\sim$ is symmetric.

Finally, to see that $\sim$ is transitive, we need to show that if $x \sim y$ and $y \sim z$, then $x \sim z$ holds. To show this, we assume the hypothesis that $x, y, z \in \mathbb{R}$ such that $x \sim y$ and $y \sim z$. Since $x \sim y$, we know that $x - y = n_1 \in \mathbb{Z}$, and since $y \sim z$ we know that $y - z = n_2 \in \mathbb{Z}$. Then we know that

$$x - z = x - z + (y - y) = (x - y) - (z - y) = n_1 - n_2.$$

Since n_1 and n_2 are integers, we know that $n_1 - n_2$, and therefore $x - z$, is also an integer. By the definition of $\sim$ this means that $x \sim z$, and therefore $\sim$ is transitive.

Since we have shown that $\sim$ is reflexive, symmetric, and transitive, we know that it is an equivalence relation. □

Claim 2. For all sets A and B, the following holds:

$$A \setminus B \subseteq A \cap B$$

Proof. We prove the statement is *false* by providing a counterexample.

Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ be sets. Then we can compute $A \setminus B$ and $A \cap B$ directly:

$$A \setminus B = \{1, 2, 3\} \setminus \{3, 4, 5\} = \{1, 2\},$$
$$A \cap B = \{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}.$$

Clearly the set $\{1, 2\}$ is not a subset of $\{3\}$, so the sets A and B above contradict the claim. □

Claim 3. For all sets A and B, the following holds:

$$x \notin B \implies x \notin A \setminus (A \setminus B) \tag{1}$$

Proof. We prove that the claim is *true*.

For convenience, we rewrite the statement of equation (1) into its equivalent contrapositive form:

$$x \in A \setminus (A \setminus B) \implies x \in B, \tag{2}$$

which happens to be equivalent to showing that $A \setminus (A \setminus B) \subseteq B$.

To prove this, we let $x \in A \setminus (A \setminus B)$; it now suffices to show that x is also in B. We show this directly by noting that by the definition of $\setminus$, we know that $x \in A$ and $x \notin A \setminus B$. Similarly, since we know that $x \notin A \setminus B$, we know that one of the following must be true: (a) $x \notin A$ or (b) $x \in B$ (if both were false that would mean that $x \in A \setminus B$). Since we already know that $x \in A$, we know that (a) must be false and therefore (b) must be true. This means that $x \in B$, and therefore the statement is true. □

Claim 4. Every undirected graph $G = (V, E)$ with $|V| \geq 2$ has two vertices with the same degree.

Proof. We prove the claim is *true* by contradiction.

Assume the claim is false. Then there must be a graph $G = (V, E)$ with $n \geq 2$ number of vertices such that every vertex $v \in V$ has a unique degree. We know that the maximum degree of any vertex is $n-1$ (when it is connected to every other vertex by an edge), and the minimum degree of any vertex is 0 (when it is connected to no other vertices). We also know that if a vertex $v \in V$ has degree $n-1$, it is impossible for a vertex $v' \in V$ to have degree 0, since v must be connected to every other vertex in V, including v'.

This means that it is only possible for a graph to have $n-1$ unique degrees, either $\{0, 1, \ldots, n-2\}$ or $\{1, 2, \ldots, n-1\}$. However, there are n vertices in the graph G, so two of them must have the same degree—a contradiction to our assumption. □

Claim 5. For all $n \in \mathbb{N}$, the following holds:

$$n^3 + 2n \text{ is divisible by } 3$$

Proof. We prove the claim is *true* by induction on n.

For the base case, consider $n = 1$. Then $n^3 + 2n = 1 + 2 = 3$ is divisible by 3, so the statement holds.

For the induction case, assume that we know that $n^3 + 2n$ is divisible by 3 for some $n \in \mathbb{N}$. It now suffices to show that $(n+1)^3 + 2(n+1)$ is also divisible by 3. By expanding the expression, we obtain

$$\begin{aligned}(n+1)^3 + 2(n+1) &= (n^3 + 3n^2 + 3n + 1) + 2n + 2 \\ &= (n^3 + 2n) + 3(n^2 + n + 1).\end{aligned}$$

By the induction hypothesis, we know that $(n^3 + 2n)$ is divisible by 3, and since the rest of the expression is a multiple of 3, the whole expression is divisible by 3. $\square$