Claim 1. The following language is not regular.

$$A = \{0^n 1 0^n \mid n \in \mathbb{N}\}$$

Proof. We prove the claim is true by contradiction. Thus we begin by assuming the opposite of the claim, i.e., that A is regular. Since A is regular, the pumping lemma applies, so there exists a pumping length p for A.

Now consider the string $w = 0^p 1 0^p$. Notice that the length of this string is $|w| = 2p + 1 \geq p$, therefore the pumping lemma tells us that w can be decomposed into three parts x, y, and z such that $w = xyz$ and that the following three statements must be true:

1. $xy^i z \in A$ for all $i \geq 0$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Since $|xy| \leq p$ and the first p symbols of $0^p 1 0^p$ are all 0s, we know that both x and y consist of only 0s to the left of the 1 in the middle. We also know that $|y| > 0$ and therefore $y = 0^k$ for some $k > 0$. Finally, we also know that $xy^2 z \in A$, and this string is

$$xy^2 z = xyyz = 0^{p+k} 1 0^p,$$

since adding an extra y increases the number of 0s on the left-hand side by $k > 0$. However, this means that the number of 0s on either side of the 1 are not equal and therefore $xy^2 z$ is not in A—this is a contradiction to the pumping lemma saying that $xy^2 z \in A$. This means our original assumption that A is regular was false, so it actually is not regular. □

Claim 2. The following language is not regular

$$B = \{p(x, y) \mid x, y \in \{0, 1\}^*\},$$

where $p(x, y)$ is the *self-delimiting pairing function* defined by

$$p(x, y) = 0^{|x|} 1 xy$$

for all strings $x, y \in \{0, 1\}^*$.

Proof. We prove that B is not regular using contradiction and the pumping lemma. Assume that B is regular, and let p be its pumping length. Now let $w = p(0^p, \epsilon) = 0^p10^p$. Since $w \in B$ and $|w| = 2p + 1 \geq p$, we know the pumping lemma applies, so $w = xyz$ for some strings x, y, and z. The pumping lemma tells us that $|xy| \leq p$ and $|y| > 0$, therefore we know that y consists of one or more 0s to the left of the first 1 in $w = 0^p10^p$. Therefore the string $xyyz = 0^{p+k}10^p$ for some $k > 0$ which is not a properly formatted self-delimited pair of strings since there are more 0s to the left of the 1. This contradicts the pumping lemma that says that $xyyz$ should be in B. □

Claim 3. The following language is not regular.

$$C = \{0^n1^m \mid n \geq m\}$$

Proof. (by contradiction) Assume that C is regular with pumping length p and let $w = 0^p1^p$. Then the pumping lemma says that $w = xyz$ satisfying $|xy| \leq p$, $|y| > 0$, and $xy^0z = xz \in C$. Therefore we know that y consists of one or more zeros on the left-hand-side and that $xz = 0^{p-k}1^p$ for some $k > 0$. Since $p - k \not\geq p$, we know that xz is not of the appropriate form to be in the language C, therefore it contradicts the pumping lemma. Thus, we can conclude that C is not regular. □