**Claim 1.** The following language is undecidable.

$$\text{TOTAL} = \{\langle M \rangle \mid M \text{ is a TM that halts on all inputs}\}$$

*Proof.* We prove the claim is true by contradiction.

Assume to the contrary that TOTAL is decidable by the Turing machine $M_{\text{TOTAL}}$. We will now construct a Turing machine that decides the following language, which we've already seen is undecidable:

$$A_{\text{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM that accepts } w\}$$

Consider the following Turing machine:

$N =$ "On input $\langle M, w \rangle$ where $M$ is a TM,

1. Construct the TM $M^*$ defined by

   $M^* =$ "On input $x$

   1. Run $M$ on $w$
   2. If $M$ accepts, then ACCEPT
   3. If $M$ rejects, then enter an infinite loop"

2. Run $M_{\text{TOTAL}}$ on $\langle M^* \rangle$
3. If it accepts, then ACCEPT; otherwise REJECT"

We now prove that $N$ is a decider for $A_{\text{TM}}$.

Consider a string $\langle M, w \rangle \in A_{\text{TM}}$. By definition, we know that $M$ accepts $w$; thus, the machine $M^*$ will accept every input strings $x$ and therefore halts on every input. This means that $M^*$ is total, and therefore $\langle M^* \rangle \in \text{TOTAL}$. Since $M_{\text{TOTAL}}$ is a decider for TOTAL, this means that it will accept the input $\langle M^* \rangle$, and therefore the Turing machine $N$ will accept $\langle M, w \rangle$.

It remains to be shown that if $\langle M, w \rangle \notin A_{\text{TM}}$ then the machine $N$ rejects. Given such an input, we know that $M$ does not accept $w$ and either will reject or get into an infinite loop. Thus, the machine $M^*$ will always loop forever, because even if $M$ halts and rejects, then $M^*$ will enter an infinite loop. This means that $M^*$ is not total and therefore $\langle M^* \rangle \notin \text{TOTAL}$. Thus, $M_{\text{TOTAL}}$ will reject $\langle M^* \rangle$, and so $N$ will reject $\langle M, w \rangle$.

Since $N$ will accept an input $\langle M, w \rangle$ if it is in $A_{\text{TM}}$ and it will reject all other strings, it is a decider for $A_{\text{TM}}$. However, we know that $A_{\text{TM}}$ is undecidable—a contradiction.

□

**Claim 2.** The following language is undecidable.

$$S = \{\langle M \rangle \mid M \text{ is a TM that accepts } w \text{ if and only if it accepts } w^R\}$$

*Proof.* It suffices to show that $A_{\text{TM}} \leq_m S$, since $A_{\text{TM}}$ is undecidable. This would then imply that $S$ is also undecidable.

To prove that $A_{\text{TM}} \leq_m S$, we must show that there is a computable function $f$ satisfying that $\langle M, w \rangle \in A_{\text{TM}}$ if and only if $f(\langle M, w \rangle) = \langle \widehat{M} \rangle \in S$. We define this function in the following way:

$$f(\langle M, w \rangle) = \langle \widehat{M} \rangle = \begin{cases} \text{On input } x, \\ \quad 1.\ \text{If } x = 01, \text{ then ACCEPT} \\ \quad 2.\ \text{If } x = 10, \text{ then} \\ \qquad \text{a. Run } M \text{ on } w \\ \qquad \text{b. If it accepts, then ACCEPT} \\ \qquad \text{c. Otherwise REJECT} \\ \quad 3.\ \text{If } x \text{ is any other string, then REJECT} \end{cases}$$

Notice that if $M$ accepts $w$, then $L(\widehat{M}) = \{01, 10\}$, and therefore $\langle \widehat{M} \rangle \in S$. Similarly, if $M$ does not accept $w$, then $L(\widehat{M}) = \{01\}$, and therefore $\langle \widehat{M} \rangle \notin S$. Thus, $\langle M, w \rangle \in A_{\text{TM}}$ if and only if $f(\langle M, w \rangle) \in S$.

We also note that $f$ is computable since it only requires a finite number of steps to transform a finite description $\langle M, w \rangle$ into the description $\langle \widehat{M} \rangle$, so it will always halt on all inputs. This means that $f$ is a reduction from $A_{\text{TM}}$ to $S$, meaning that $A_{\text{TM}} \leq_m S$. This immediately implies that $S$ is undecidable. □